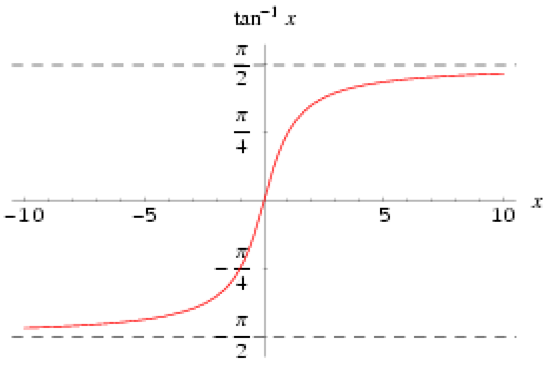
Inverse Tangent

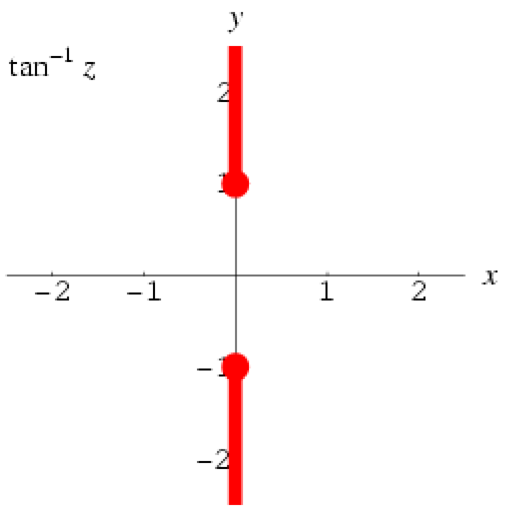
The inverse tangent is the multivalued function $\tan^{-1} z$ (Zwillinger 1995, p. 465), also denot
79; Harris and Stocker 1998, p. 311; Jeffrey 2000, p. 124) or $\operatorname{arctg} z$ (Spanier and Oldham
p. 208; Jeffrey 2000, p. 127), that is the inverse function of the tangent. The variants Arct
1997, p. 70) and $\operatorname{Tan}^{-1} z$ are sometimes used to refer to explicit principal values of the inve
always made (e.g,. Zwillinger 1995, p. 466).



The inverse tangent function $\tan^{-1} x$ is plotted above along the real axis.

Worse yet, the notation $\arctan z$ is sometimes used for the principal value, with $\operatorname{Arctan} z$ bei
(Abramowitz and Stegun 1972, p. 80). Note that in the notation $\tan^{-1} z$ (commonly used in
worldwide), $\tan z$ denotes the tangent and -1 the inverse function, *not* the multiplicative inv

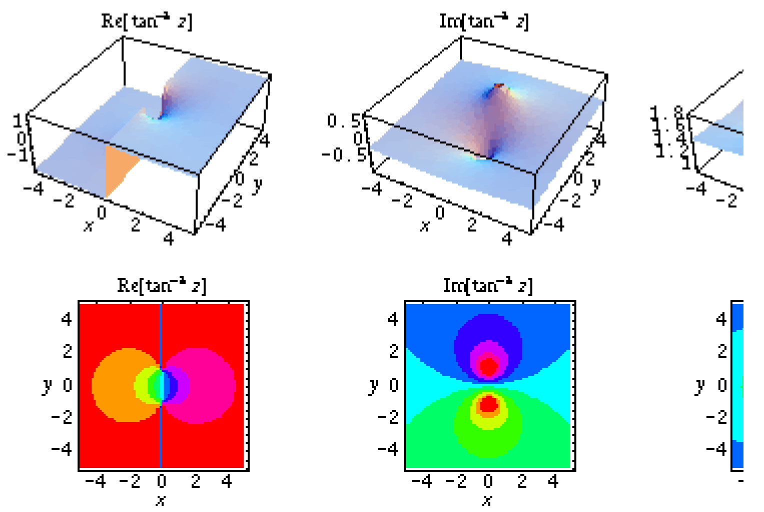
The principal value of the inverse tangent is implemented as `ArcTan`[*z*] in *Mathematica*. In t
(*double x*).



The inverse tangent is a multivalued function and hence requires a branch cut in the comple
places at $(-i\infty, -i]$ and $[i, i\infty)$. This follows from the definition of $\tan^{-1} z$ as

$$\tan^{-1} z = \frac{1}{2} i [\ln(1 - i z) - \ln(1 + i z)].$$

In *Mathematica* (and in this work), this branch cut definition determines the range of $\tan^{-1}$ z taken, however, as other branch cut definitions can give different ranges (most commonly,



	Min	Max
Re	-5	5
Im	-5	5

Replot

Register for Unlimited Interacti

The inverse tangent function $\tan^{-1} z$ is plotted above in the complex plane.

$\tan^{-1} z$ has the special values

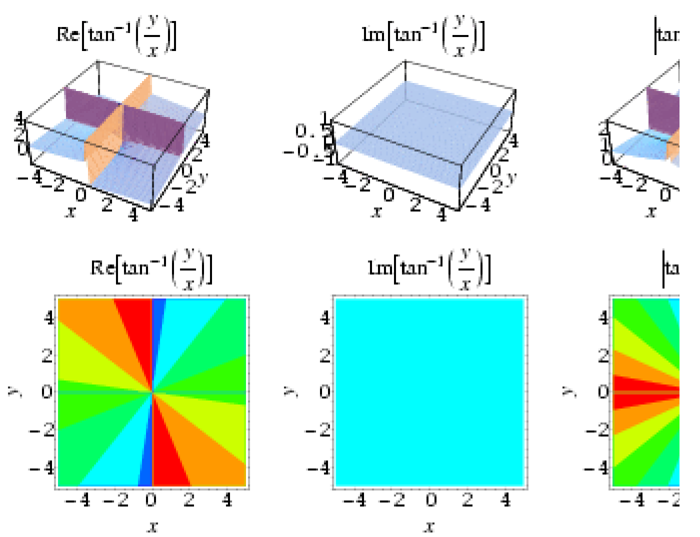
$$\begin{aligned}
\tan^{-1}(-\infty) &= -\tfrac{1}{2}\pi \\
\tan^{-1}(-i) &= -i\infty \\
\tan^{-1} 0 &= 0 \\
\tan^{-1} i &= i\infty \\
\tan^{-1}\infty &= \tfrac{1}{2}\pi.
\end{aligned}$$

The derivative of $\tan^{-1} z$ is

$$\frac{d}{dz}\tan^{-1} z = \frac{1}{1+z^2}$$

and the indefinite integral is

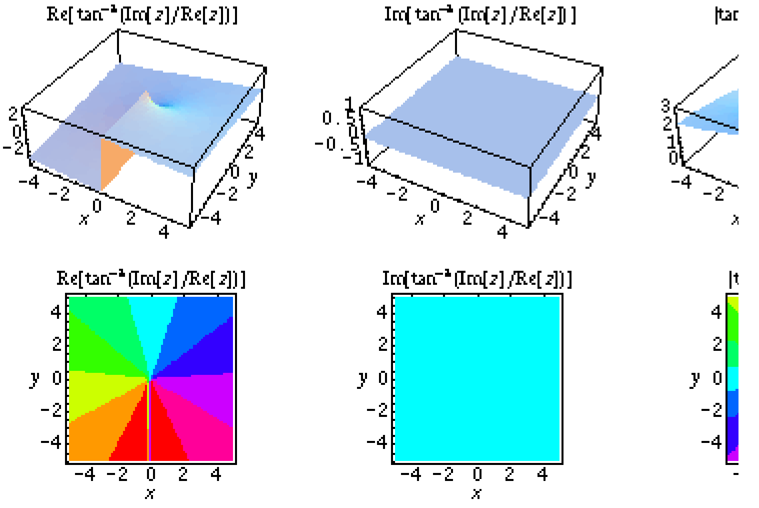
$$\int \tan^{-1} z\, dz = z\tan^{-1} z - \tfrac{1}{2}\ln(1+z^2) + C.$$



The complex argument of a complex number $z = x + i\,y$ is often written as

$$\theta = \tan^{-1}\left(\frac{y}{x}\right),$$

where θ, sometimes also denoted ϕ, corresponds to the counterclockwise angle from the po
$x = \cos\theta$ and $y = \sin\theta$. Plots of $\tan^{-1}(y/x)$ are illustrated above for real values of x and y.



	Min	Max
Re	-5	5
Im	-5	5

Replot

Register for Unlimited Interacti

A special kind of inverse tangent that takes into account the quadrant in which z lies and is
x), the GNU C library command atan2(*double y*, *double x*), and the *Mathematica* command
range $-\pi < \theta \leq \pi$. In the degenerate case when $x = 0$,

$$\phi = \begin{cases} -\frac{1}{2}\pi & \text{if } y < 0 \\ \text{undefined} & \text{if } y = 0 \\ \frac{1}{2}\pi & \text{if } y > 0. \end{cases}$$

The usual $\tan^{-1} z$ has the Maclaurin series of

$$\tan^{-1} z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{2n+1}$$
$$= z - \frac{1}{3}z^3 + \frac{1}{5}z^5 - \frac{1}{7}z^7 + \ldots$$

(Sloane's A033999 and A005408). A more rapidly converging form due to Euler is given by

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{2^{2n}(n!)^2}{(2n+1)!} \frac{x^{2n+1}}{(1+x^2)^{n+1}}$$

for real x (Castellanos 1988). This is related to the formula of Euler given by

$$\tan^{-1} x = \frac{y}{x}\left(1 + \frac{2}{3}y + \frac{2\cdot 4}{3\cdot 5}y^2 + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}y^3 + \ldots\right),$$

where

$$y \equiv \frac{x^2}{1+x^2}.$$

The inverse tangent formulas are connected with many interesting approximations to pi

$$\tan^{-1}(1+x) = \frac{1}{4}\pi + \frac{1}{2}x - \frac{1}{4}x^2 + \frac{1}{12}x^3 - \frac{1}{40}x^5 + \frac{1}{48}x^6 - \frac{1}{112}x^7 + \ldots$$

(Sloane's A075553 and A075554).

The inverse tangent satisfies

$$\tan^{-1} z = \cot^{-1}\left(\frac{1}{z}\right)$$

for $z \neq 0$,

$$\tan^{-1} z = -\tan^{-1}(-z)$$

for all complex z,

$$\begin{aligned}\tan^{-1} x &= \frac{1}{2}\pi - \cos^{-1}\left(\frac{x}{\sqrt{x^2+1}}\right) \\ &= \sin^{-1}\left(\frac{x}{\sqrt{x^2+1}}\right) \\ &= \csc^{-1}\left(\frac{\sqrt{x^2+1}}{x}\right)\end{aligned}$$

for all real x, where equality for the last equation is understood to be in the limit as $x \to 0$,

$$\begin{aligned}\tan^{-1} x &= \begin{cases} -\frac{1}{2}\pi - \tan^{-1}\left(\frac{1}{x}\right) & \text{for } x < 0 \\ \frac{1}{2}\pi - \tan^{-1}\left(\frac{1}{x}\right) & \text{for } x > 0 \end{cases} \\ &= \begin{cases} -\frac{1}{2}\pi + \cot^{-1}(-x) & \text{for } x < 0 \\ \frac{1}{2}\pi + \cot^{-1}(-x) & \text{for } x > 0 \end{cases} \\ &= \begin{cases} -\frac{1}{2}\pi - \cot^{-1} x & \text{for } x < 0 \\ \frac{1}{2}\pi - \cot^{-1} x & \text{for } x > 0 \end{cases} \\ &= \begin{cases} -\cos^{-1}\left(\frac{1}{\sqrt{x^2+1}}\right) & \text{for } x < 0 \\ \cos^{-1}\left(\frac{1}{\sqrt{x^2+1}}\right) & \text{for } x > 0 \end{cases} \\ &= \begin{cases} -\sec^{-1}\left(\sqrt{x^2+1}\right) & \text{for } x < 0 \\ \sec^{-1}\left(\sqrt{x^2+1}\right) & \text{for } x > 0. \end{cases}\end{aligned}$$

In terms of the hypergeometric function,

$$\tan^{-1} z = z\, {}_2F_1\left(1, \tfrac{1}{2}; \tfrac{3}{2}; -z^2\right)$$

for complex z, and

$$\tan^{-1} x = \frac{x}{1+x^2} \, {}_2F_1\left(1, 1; \tfrac{3}{2}; \frac{x^2}{1+x^2}\right)$$

for real x (Castellanos 1988).

Castellanos (1986, 1988) also gives some curious formulas in terms of the Fibonacci numbe

$$\begin{aligned} \tan^{-1} x &= \sum_{n=0}^{\infty} \frac{(-1)^n F_{2n+1} t^{2n+1}}{5^n (2n+1)} \\ &= 5 \sum_{n=0}^{\infty} \frac{(-1)^n F_{2n+1}^2}{(2n+1)\left(u+\sqrt{u^2+1}\right)^{2n+1}} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n 5^{n+2} F_{2n+1}^3}{(2n+1)\left(v+\sqrt{v^2+5}\right)^{2n+1}}, \end{aligned}$$

where

$$\begin{aligned} t &\equiv \frac{2x}{1+\sqrt{1+\frac{4x^2}{5}}} \\ u &\equiv \frac{5}{4x}\left(1+\sqrt{1+\tfrac{24}{25}x^2}\right), \end{aligned}$$

and v is the largest positive root of

$$8 x v^4 - 100 v^3 - 450 x v^2 + 875 v + 625 x = 0.$$

The inverse tangent satisfies the addition formula

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

for $-1 < x, y < 1$, as well as the more complicated formulas

$$\begin{aligned} \tan^{-1}\left(\frac{1}{a-b}\right) &= \tan^{-1}\left(\frac{1}{a}\right) + \tan^{-1}\left(\frac{b}{a^2 - ab + 1}\right) \\ \tan^{-1}\left(\frac{1}{a}\right) &= 2\tan^{-1}\left(\frac{1}{2a}\right) - \tan^{-1}\left(\frac{1}{4a^3 + 3a}\right) \\ \tan^{-1}\left(\frac{1}{p}\right) &= \tan^{-1}\left(\frac{1}{p+q}\right) + \tan^{-1}\left(\frac{q}{p^2 + pq + 1}\right), \end{aligned}$$

the latter of which was known to Euler. Another interesting inverse tangent identity attribut Lehmer (1938b; Bromwich 1965, Castellanos 1988ab) is

$$\tan^{-1}(p+r) + \tan^{-1}(p+q) - \tan^{-1} p = \tfrac{1}{2}\pi,$$

where

$$1 + p^2 = q r$$

and $p, q, r > 0$.

The inverse tangent has continued fraction representations

$$\tan^{-1} x = \cfrac{x}{1 + \cfrac{x^2}{3 + \cfrac{4x^2}{5 + \cfrac{9x^2}{7 + \cfrac{16x^2}{9 + \dots}}}}}$$

(Lambert 1770; Lagrange 1776; Wall 1948, p. 343; Olds 1963, p. 138) and

$$\tan^{-1} x = \cfrac{x}{1 + \cfrac{x^2}{3 - x^2 + \cfrac{9x^2}{5 - 3x^2 + \cfrac{25x^2}{7 - 5x^2 + \dots}}}}$$

due to Euler and sometimes known as Euler's continued fraction (Borwein *et al.* 2004, p. 30

To find $\tan^{-1} x$ numerically, the following arithmetic-geometric mean-like algorithm can be

$$\begin{aligned} a_0 &= (1 + x^2)^{-1/2} \\ b_0 &= 1. \end{aligned}$$

Then compute

$$\begin{aligned} a_{i+1} &= \tfrac{1}{2}(a_i + b_i) \\ b_{i+1} &= \sqrt{a_{i+1} b_i}, \end{aligned}$$

and the inverse tangent is given by

$$\tan^{-1} x = \lim_{n \to \infty} \frac{x}{\sqrt{1 + x^2}\, a_n}$$

(Acton 1990).

An inverse tangent $\tan^{-1} n$ with integral n is called reducible if it is expressible as a finite su

$$\tan^{-1} n = \sum_{k=1} f_k \tan^{-1} n_k,$$

where f_k are positive or negative integers and n_i are integers $< n$. $\tan^{-1} m$ is reducible iff a
the prime factors of $1 + n^2$ for $n = 1, \ldots, m - 1$. A second necessary and sufficient condition
less than $2m$. Equivalent to the second condition is the statement that every Gregory numb
a sum in terms of t_ms for which m is a Størmer number (Conway and Guy 1996). To find th

$$\arg(1 + i\,n) = \arg \prod_{k=1} (1 + n_k\, i)^{f_k},$$

so the ratio

$$r = \frac{\prod_{k=1} (1 + n_k\, i)^{f_k}}{1 + i\,n}$$

is a rational number. Equation (50) can also be written

$$r^2 (1 + n^2) = \prod_{k=1} (1 + n_k^2)^{f_k}.$$

Writing (□) in the form

$$\tan^{-1} n = \sum_{k=1} f_k \tan^{-1} n_k + f \tan^{-1} 1$$

allows a direct conversion to a corresponding inverse cotangent formula

$$\cot^{-1} n = \sum_{k=1} f_k \cot^{-1} n_k + c \cot^{-1} 1,$$

where

$$c = 2 - f - 2 \sum_{k=1} f_r.$$

Todd (1949) gives a table of decompositions of $\tan^{-1} n$ for $n \leq 342$. Conway and Guy (1996)
numbers.

Arndt and Gosper give the remarkable inverse tangent identity

$$\sin\left(\sum_{k=1}^{2n+1} \tan^{-1} a_k\right) = \frac{(-1)^n}{2n+1} \frac{\sum_{k=1}^{2n+1} \prod_{j=1}^{2n+1} \left[a_j - \tan\left(\frac{\pi (j-k)}{2n+1}\right)\right]}{\sqrt{\prod_{j=1}^{2n+1} (a_j^2 + 1)}}.$$

There is an amazing set of BBP-type formulas for $\tan^{-1}(4/5)$

$$\tan^{-1}\left(\frac{4}{5}\right) = \frac{1}{131072} \sum_{k=0}^{\infty} \frac{1}{1048576^k} \Big[\frac{262144}{40k+2} - \frac{163840}{40k+5} - \frac{65536}{40k+6} + \frac{16384}{40k+10} - \frac{409\ldots}{40k+\ldots} \ldots \frac{256}{40k+22} + \frac{160}{40k+25} + \frac{64}{40k+26} - \frac{16}{40k+30} + \frac{\ldots}{40\ldots}$$

$$= \frac{1}{131072} \sum_{k=0}^{\infty} \frac{1}{1048576^k} \Big[\frac{393216}{40k+4} + \frac{163840}{40k+5} - \frac{131072}{40k+6} - \frac{163840}{40k+8} + \frac{24576\ldots}{40k+1\ldots} \ldots \frac{10240}{40k+16} - \frac{1024}{40k+20} - \frac{512}{40k+22} - \frac{640}{40k+24} - \frac{\ldots}{40\ldots} \ldots \frac{32}{40k+30} - \frac{40}{40k+32} + \frac{15}{40k+35} + \frac{6}{40k+36} - \ldots$$

$$= \frac{1}{131072} \sum_{k=0}^{\infty} \frac{1}{1048576^k} \Big[\frac{262144}{40k+1} - \frac{262144}{40k+3} - \frac{65536}{40k+5} - \frac{327680}{40k+6} + \frac{65536}{40k+7\ldots} \ldots \frac{40960}{40k+10} - \frac{16384}{40k+11} - \frac{4096}{40k+13} - \frac{20480}{40k+14} - \frac{1\ldots}{40\ldots} \ldots \frac{1024}{40k+17} - \frac{1024}{40k+19} - \frac{2560}{40k+20} - \frac{256}{40k+21} - \frac{\ldots}{40\ldots} \ldots \frac{640}{40k+24} + \frac{64}{40k+25} - \frac{64}{40k+27} - \frac{16}{40k+29} - \frac{\ldots}{40\ldots} \ldots \frac{40}{40k+32} + \frac{4}{40k+33} + \frac{16}{40k+35} - \frac{1}{40k+37} - \frac{\ldots}{40\ldots}$$

$$=$$

$$\frac{1}{262144}\sum_{k=0}^{\infty}\frac{1}{1048576^k}\Bigg[$$

$$\frac{262144}{40k+3}+\frac{262144}{40k+4}+\frac{131072}{40k+6}-\frac{65536}{40k+7}+\frac{81920}{40k+1}$$

$$\frac{8192}{40k+14}-\frac{4096}{40k+15}+\frac{1024}{40k+19}+\frac{1024}{40k+20}+$$

$$\frac{512}{40k+22}-\frac{256}{40k+23}+\frac{64}{40k+27}+\frac{64}{40k+28}-\ 40$$

$$\frac{16}{40k+31}+\frac{4}{40k+35}+\frac{4}{40k+36}+\frac{2}{40k+38}-\ 40$$

the finding one of which is a given as a problem by Bailey *et al.* (2006, p. 225).

SEE ALSO: Euler's Machin-Like Formula, Gauss's Machin-Like Formula, Inverse Cotangent, Ir
Formula, Machin-Like Formulas, Tangent. [Pages Linking Here]

RELATED WOLFRAM SITES: http://functions.wolfram.com/ElementaryFunctions/ArcTan

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